Critical points and classification

Let $f(x, y) = ax^2 + 2bxy + y^2$, where a and b are fixed.

Assuming $a \neq b^2$, verify that $(0, 0)$ is a critical point and classify it based on the values of a and b. Analyze the case where $a = b^2$ to determine the critical points of f.

Solution

First, we find the critical points by computing the partial derivatives and setting them equal to zero.

Compute the partial derivatives of f:

- $f_x = \dfrac{\partial f}{\partial x} = 2ax + 2by$
- $f_y = \dfrac{\partial f}{\partial y} = 2bx + 2y$

Set the partial derivatives equal to zero:

$$\begin{cases} 2ax + 2by = 0 & (1) \\ 2bx + 2y = 0 & (2) \end{cases}$$

Simplify:

$$\begin{cases} ax + by = 0 & (1) \\ bx + y = 0 & (2) \end{cases}$$

Solve for y from equation (2):

$$y = -bx$$

Substitute into equation (1):

$$ax + b(-bx) = 0 \implies ax - b^2x = 0$$

Factorize:

$$(a - b^2)x = 0$$

Case 1: $a \neq b^2$
Since $a - b^2 \neq 0$, it follows that:

$$x = 0$$

Thus:

$$y = -bx = 0$$

Conclusion: The only critical point is $(0,0)$.

Now, we classify the critical point using the second derivative test. Compute the second-order partial derivatives:

- $f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2a$
- $f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2$
- $f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 2b$

Compute the Hessian determinant at $(0,0)$:

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (2a)(2) - (2b)^2 = 4a - 4b^2 = 4(a - b^2)$$

Depending on the sign of D and f_{xx}, classify the critical point:

- If $D > 0$ and $f_{xx} > 0$, it is a **local minimum**.

- If $D > 0$ and $f_{xx} < 0$, it is a **local maximum**.
- If $D < 0$, it is a **saddle point**.

Analysis based on the values of a and b:

- **If $a - b^2 > 0$:**
  - $D = 4(a - b^2) > 0$
  - $f_{xx} = 2a$
    - If $a > 0$, then $f_{xx} > 0 \implies$ **Local minimum at** $(0, 0)$.
    - If $a < 0$, then $f_{xx} < 0 \implies$ **Local maximum at** $(0, 0)$.
- **If $a - b^2 < 0$:**
  - $D = 4(a - b^2) < 0$
  - **Saddle point at** $(0, 0)$.

Case 2: $a = b^2$
When $a - b^2 = 0$, the Hessian determinant $D = 0$, so the second derivative test is inconclusive.

Analyze the critical points in this case:
From the equations:

$$\begin{cases} (a - b^2)x = 0 \\ y = -bx \end{cases}$$

Since $a - b^2 = 0$, the equation $(a - b^2)x = 0$ is satisfied for any value of x. Thus, all points on the line $y = -bx$ are critical points.

Evaluate f on this line:

$$f(x, -bx) = ax^2 + 2bx(-bx) + (-bx)^2 = ax^2 - 2b^2x^2 + b^2x^2 = (a - b^2)x^2 = 0$$

Since $a = b^2$, we find $f(x, -bx) = 0$ for all x.
Conclusion: When $a = b^2$, all points on the line $y = -bx$ are critical points, and f is constant and equal to zero along this line.